Strategic games

- Strategic games are theoretical models of (social, economic, political) strategic interactions.
- They can be represented formally and solved, game theory is a branch of mathematics.

Strategic games

- Assumptions:
  - Well-ordered preferences (complete, reflexive, transitive)
  - For some concepts also continuous (i.e., essentially numerical) preferences

- Advantages:
  - Straightforward representation of strategic interactions, direct interdependence, heterogenous agents, etc.
  - Analytical solution concepts

Normal form games

A normal form game is a game in which perfectly and completely informed agents choose simultaneously from a well-defined set of strategies.

Representation:
s_{R1}: Strategy 1, agent R
s_{R2}: Strategy 2, agent R
s_{C1}: Strategy 1, agent C
s_{C2}: Strategy 2, agent C
$\Pi()$: Payoff function, e.g.
$\Pi_R(s_{R1}, s_{C1})$: Payoff of agent R resulting from strategy combination s_{R1} - s_{C1}.

Normal form games

		p s_{C1}	$1-p$ s_{C2}
q	s_{R1}	$\Pi_C(s_{R1}, s_{C1})$ $\Pi_R(s_{R1}, s_{C1})$	$\Pi_C(s_{R1}, s_{C2})$ $\Pi_R(s_{R1}, s_{C2})$
$1-q$	s_{R2}	$\Pi_C(s_{R2}, s_{C1})$ $\Pi_R(s_{R2}, s_{C1})$	$\Pi_C(s_{R2}, s_{C2})$ $\Pi_R(s_{R2}, s_{C2})$

Normal form games: Row player's payoffs

		p s_{C1}	$1-p$ s_{C2}
q	s_{R1}	$\Pi_C(s_{R1}, s_{C1})$ $\Pi_R(s_{R1}, s_{C1})$	$\Pi_C(s_{R1}, s_{C2})$ $\Pi_R(s_{R1}, s_{C2})$
$1-q$	s_{R2}	$\Pi_C(s_{R2}, s_{C1})$ $\Pi_R(s_{R2}, s_{C1})$	$\Pi_C(s_{R2}, s_{C2})$ $\Pi_R(s_{R2}, s_{C2})$

Normal form games: Outcomes

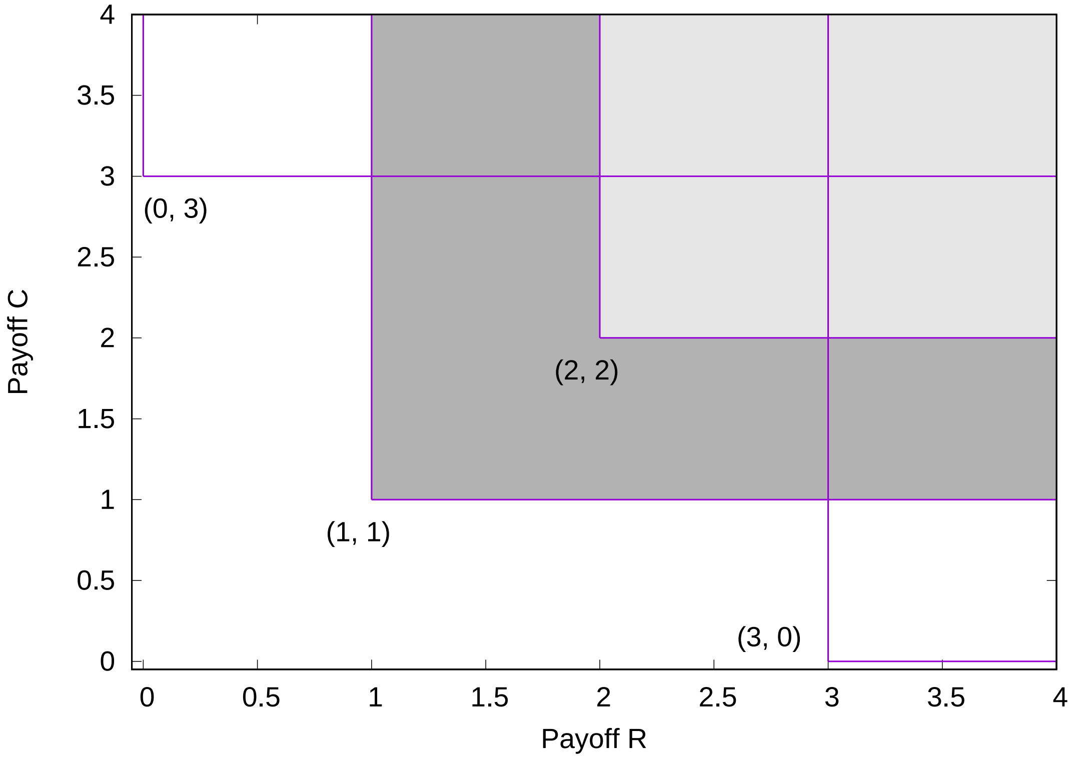
		p	$1-p$
		s_{C1}	s_{C2}
q	s_{R1}	2 2	3 0
$1-q$	s_{R2}	0 3	1 1

- Which is the best possible outcome?

Normal form games: Pareto optimality

Payoff-valuation independent
optimality concept
(impossibility of improvement
without redistribution)

		p s_{C1}	$1-p$ s_{C2}
q	s_{R1}	2 2	3 0
$1-q$	s_{R2}	0 3	**1** **1**



Normal form games: Nash equilibria

Combination of mutual best answer strategies

		p s_{C1}	$1-p$ s_{C2}
q	s_{R1}	2 2	3 0
$1-q$	s_{R2}	0 3	**1** **1**

Mixed strategies

		p	$1-p$
		s_{C1}	s_{C2}
q	s_{R1}	-1 1	1 -1
$1-q$	s_{R2}	1 -1	-1 1

- Consider this game
- (Note that it is not symmetric)
- Does it have a Nash equilibrium?

Mixed strategies: Example

The condition for mixed-strategy NE is indifference between all combinations q, p:

$\frac{\partial \Pi_R}{\partial p} = 0$ and $\frac{\partial \Pi_C}{\partial q} = 0$

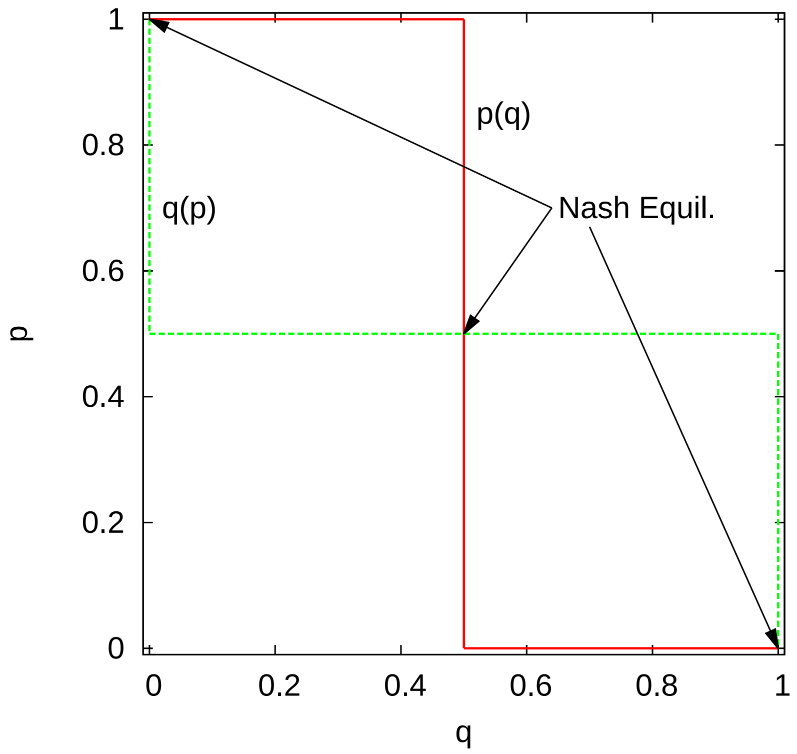
$$\Pi_R = 4pq + 2p(1-q) + 5q(1-p) + (1-p)(1-q)$$

$$\Pi_R = -2pq + p + 4q + 1$$
$$\frac{\partial \Pi_R}{\partial p} = -2q + 1 =! 0$$
$$q^* = \frac{1}{2}.$$

Because of symmetry, $p^* = q^* = \frac{1}{2}$. The mixed strategy NE is $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$-$\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$.

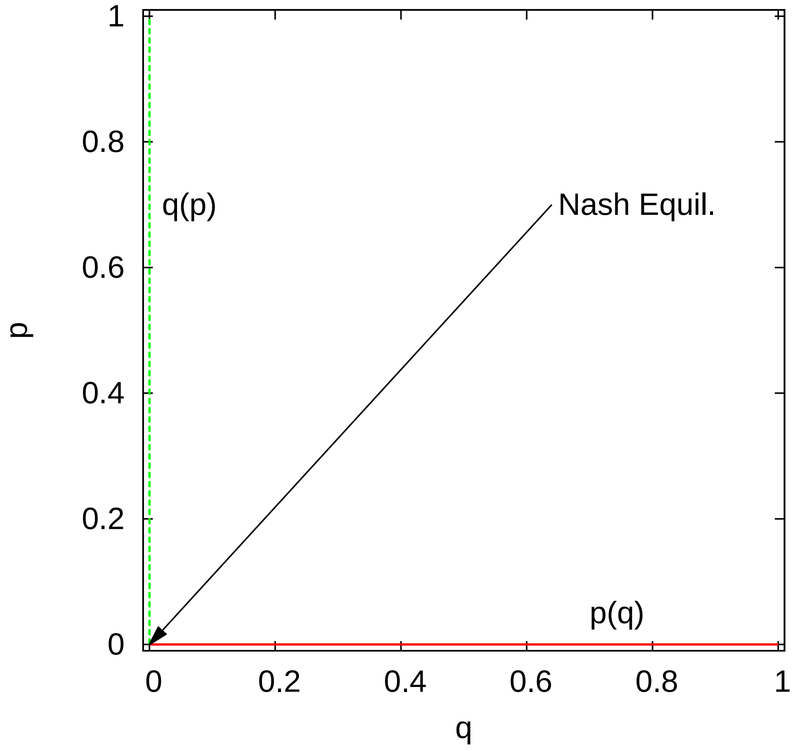
		p s_{C1}	$1-p$ s_{C2}
q	s_{R1}	4 4	**5** **2**
$1-q$	s_{R2}	**2** **5**	1 1

Specific games: Anti-coordination game



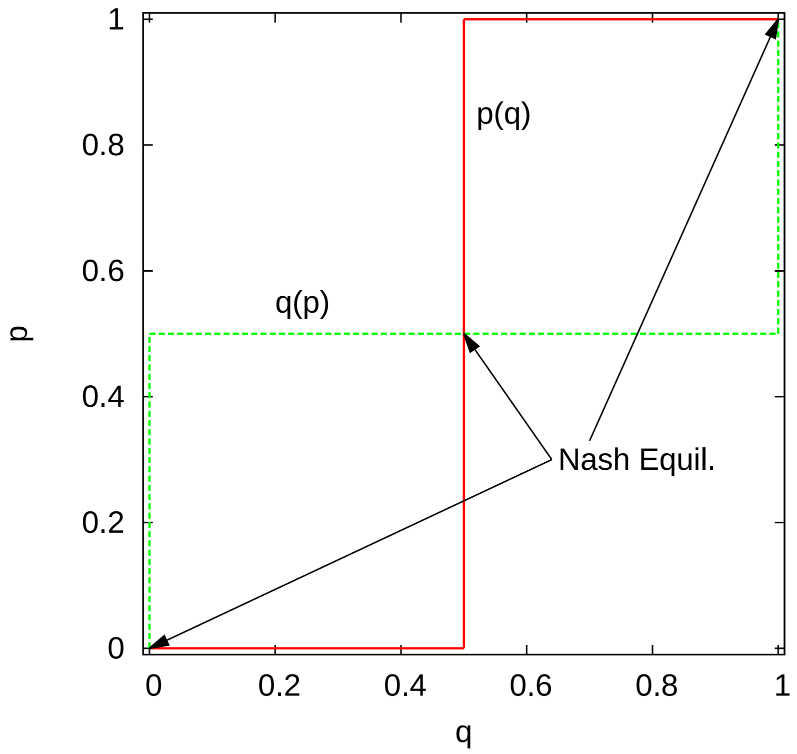
		p s_{C1}	$1-p$ s_{C2}
q	s_{R1}	4 4	**5** **2**
$1-q$	s_{R2}	**2** **5**	1 1

Specific games: Prisoners' Dilemma



		p s_{C1}	$1-p$ s_{C2}
q	s_{R1}	2 2	3 0
$1-q$	s_{R2}	0 3	**1** **1**

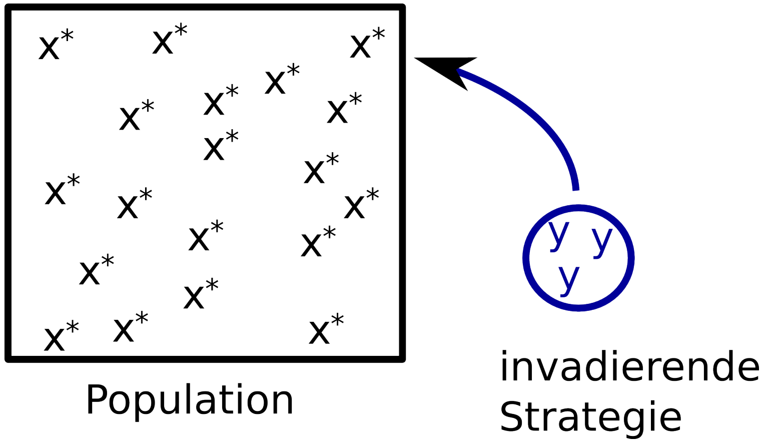
Specific games: Coordination Game



		p s_{C1}	$1-p$ s_{C2}
q	s_{R1}	**2** **2**	2 1
$1-q$	s_{R2}	1 2	**3** **3**

Evolutionary game theory

- Consider a large number of agents ...
- that are paired randomly to play simple symmetric 2-person normal form games.
- This is repeated many times with the same game structure but different opponents.
- Agents play a fixed strategy (no flexibility, no choice, no rationality).
- Strategies with a higher average payoff (compared to the population average) gain higher usage shares, others are lose usage shares.



Evolutionary game theory

x^*, y, x: Strategies

ε: Arbitrarily small share (hence also $1-\varepsilon$: arbitrarily large share

$\mathcal{A}$: Evolutionary game matrix (payoff matrix of the row player)

		p s_1	$1-p$ s_2
q	s_1	a a	c d
$1-q$	s_2	d c	b b

becomes $\mathcal{A} = \begin{pmatrix} a & d \\ c & b \end{pmatrix}$.

The expected payoff of mixed strategy $x^* = \begin{pmatrix} p \\ 1-p \end{pmatrix}$ against strategy $y = \begin{pmatrix} q \\ 1-q \end{pmatrix}$ is computed as

$$E(\Pi(x^*, y)) = x^{*T}\mathcal{A}y = \begin{pmatrix} p & 1-p \end{pmatrix} \begin{pmatrix} a & d \\ c & b \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix}$$

Evolutionary stability

$$= \left(\begin{array}{cc} ap + d(1-p) & cp + b(1-p) \end{array} \right) \left(\begin{array}{c} q \\ 1-q \end{array} \right) = (ap+d(1-p))q+(cp+b(1-p))(1-q)$$

$$E(\Pi(x^*, P)) > E(\Pi(y, P)) \qquad \forall y \neq x^*$$
$$x^{*T} \mathcal{A} P > y^T \mathcal{A} P$$

Substituting the population shares yields:

$$x^{*T} \mathcal{A}((1-\varepsilon)x^* + \varepsilon y) > y^T \mathcal{A}((1-\varepsilon)x^* + \varepsilon y)$$

$$x^{*T} \mathcal{A} x^* - \varepsilon x^{*T} \mathcal{A} x^* + \varepsilon x^{*T} \mathcal{A} y > y^T \mathcal{A} x^* - \varepsilon y^T \mathcal{A} x^* + \varepsilon y^T \mathcal{A} y$$

1st condition of evolutionary stability:

$$x^{*T} \mathcal{A} x^* \geq y^T \mathcal{A} x^*$$

Evolutionary stability

Case 1: Condition holds strictly, $x^{*T}\mathcal{A}x^* > y^T\mathcal{A}x^*$.
Case 2: Condition holds marginally, $x^{*T}\mathcal{A}x^* = y^T\mathcal{A}x^*$. In this case, $y^T\mathcal{A}x^*$ may be substituted for $x^{*T}\mathcal{A}x^*$ in the above equation.

$$y^T\mathcal{A}x^* - \varepsilon y^T\mathcal{A}x^* + \varepsilon x^{*T}\mathcal{A}y > y^T\mathcal{A}x^* - \varepsilon y^T\mathcal{A}x^* + \varepsilon y^T\mathcal{A}y$$

$$\varepsilon x^{*T}\mathcal{A}y > \varepsilon y^T\mathcal{A}y$$

Eliminating ε yields the **2nd condition of evolutionary stability**:

$$x^{*T}\mathcal{A}y > y^T\mathcal{A}y$$

Replicator models

- Replicator models are dynamical models that express the development of population shares explicitly.
- These population shares may refer to populations of firms (or other economic entities) ...
- ... the fitness of which is characterised by their economic success (profits, payoff, ...)
- ... which is determined by strategies (or technologies or routines) employed by the agent.
- Pactically, firms with the same strategy and fitness are not distinguishable in the model. They can be written as a compound entity.
- The resulting formal models are time-invariant dynamical systems.
- Models with constant or non-constant fitness are conceivable (e.g. if the fitness depends on the population share.
- There are stockastic extensions.

Replicator dynamics

- Terms:
  - p_i: Population share of group i
  - f_i: Evolutionary fitness of group i
  - $\phi = \sum_i p_i f_i$: Average fitness of the population

- Examples for development equations commonly used in replicator dynamics are

$$\frac{dp_i}{dt} = p_i(f_i - \phi) \qquad\qquad p_{i,t+1} = p_{i,t}\frac{f_{i,t}}{\phi_t}$$

$$p_{i,t+1} = p_{i,t} + p_{i,t}(f_{i,t} - \phi_t)$$

- Commonly $\frac{\partial dp_i/dt}{\partial f_i} > 0$, $\frac{\partial dp_i/dt}{\partial \phi} < 0$

Example: Replicator dynamics

- n agents
- ... navigate a web of narrow streets
- ... encounter each other frequently, have to decide between an non-aggressive strategy (slowing down) and an aggressive strategy (not slowing down)
- ... have a fixed strategy in the beginning.
- Agents have a base payoff of 5 on average for every encounter (This leads to total payoffs being positive, which simplifies equations without changing the structure of the system.)
- Encounters of two aggressive agents result in losses of 4 for both agents.
- Encounters of two non-aggressive agents result in losses of 1 for both agents.
- Encounters of an aggressive and a non-aggressive agent result in a loss of 3 for the non-aggressive agent.
- Agents monitor the performance of strategies and consider changing to more successful strategies.
- The shares of a strategy change in the same proportion as the difference of the strategies' average payoff compared to the population average payoff.
- Which predictions can be made about the system?

Example - Solution

- Given: Population shares $p_i(t)$ of the strategies $i = 1, 2$ (at time t)
- Given: Payoff of strategies Π_i, average payoff in the population $\phi(t) = \sum_i \Pi_i p_i(t)$
- Given: Development equation

$$\frac{dp_i(t)}{dt} = p_i(t)(\Pi_i - \overline{\Pi})$$

- Define: evolutionary fitness of strategies $f_i = \Pi_i$, average fitness $\phi_t = \sum_i f_i p_i(t)$
- Replicator equation simplifies to

$$\frac{dp_i(t)}{dt} = p_i(t)(f_i - \phi(t)) \tag{1}$$

Example - Solution

- The model is based on an evolutionary game, hence the payoff distribution can be modeled using this game; game matrix:

		Agent B	
		aggressive (x_1)	non-aggressive (x_2)
Agent A	aggressive (x_1)	1 1	2 5
	non-aggressive (x_2)	5 2	4 4

evolutionary game matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix}$$

- Expected payoffs are computed using matrix multiplication (e.g. x_1 against x_2) $x_1^T \mathcal{A} x_2$

Example - Solution

- As agents are, on average, playing against the current population shares $p(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} = \begin{pmatrix} p_1(t) \\ 1 - p_1(t) \end{pmatrix}$
- ... this yields fitness $f_i = x_i \mathcal{A} p(t)$
- ... and average fitness $\phi(t) = p_1(t) x_1 \mathcal{A} p(t) + (1 - p_1(t)) x_2 \mathcal{A} p(t)$
- ... and for replicator equation (1)

$$\frac{dp_1(t)}{dt} = p_1(x_1 \mathcal{A} p(t) - p_1 x_1 \mathcal{A} p(t) - (1 - p_1) x_2 \mathcal{A} p(t)) \tag{2}$$

$$\frac{dp_1(t)}{dt} = p_1(1 - p_1)(x_1 \mathcal{A} p(t) - x_2 \mathcal{A} p(t)) \tag{3}$$

Example - Solution

$$\frac{dp_1(t)}{dt} = p_1(1-p_1)((5-4p_1)-(4-2p_1)) \tag{4}$$

$$\frac{dp_1(t)}{dt} = p_1(1-p_1)(1-2p_1) = 2p_1^3 - 3p_1^2 + p_1 \tag{5}$$

- Since $p_2 = 1 - p_1$, the system is completely determined by p_1, thus one-dimensional
- Solutions

$$p^*_{1,1} = 0 \qquad p^*_{1,2} = 1 \qquad p^*_{1,3} = 0,5$$

- Only element of Jacobian

$$\frac{\partial \frac{dp_1}{dt}}{\partial p_1} = 6p_1^2 - 6p_1 + 1$$

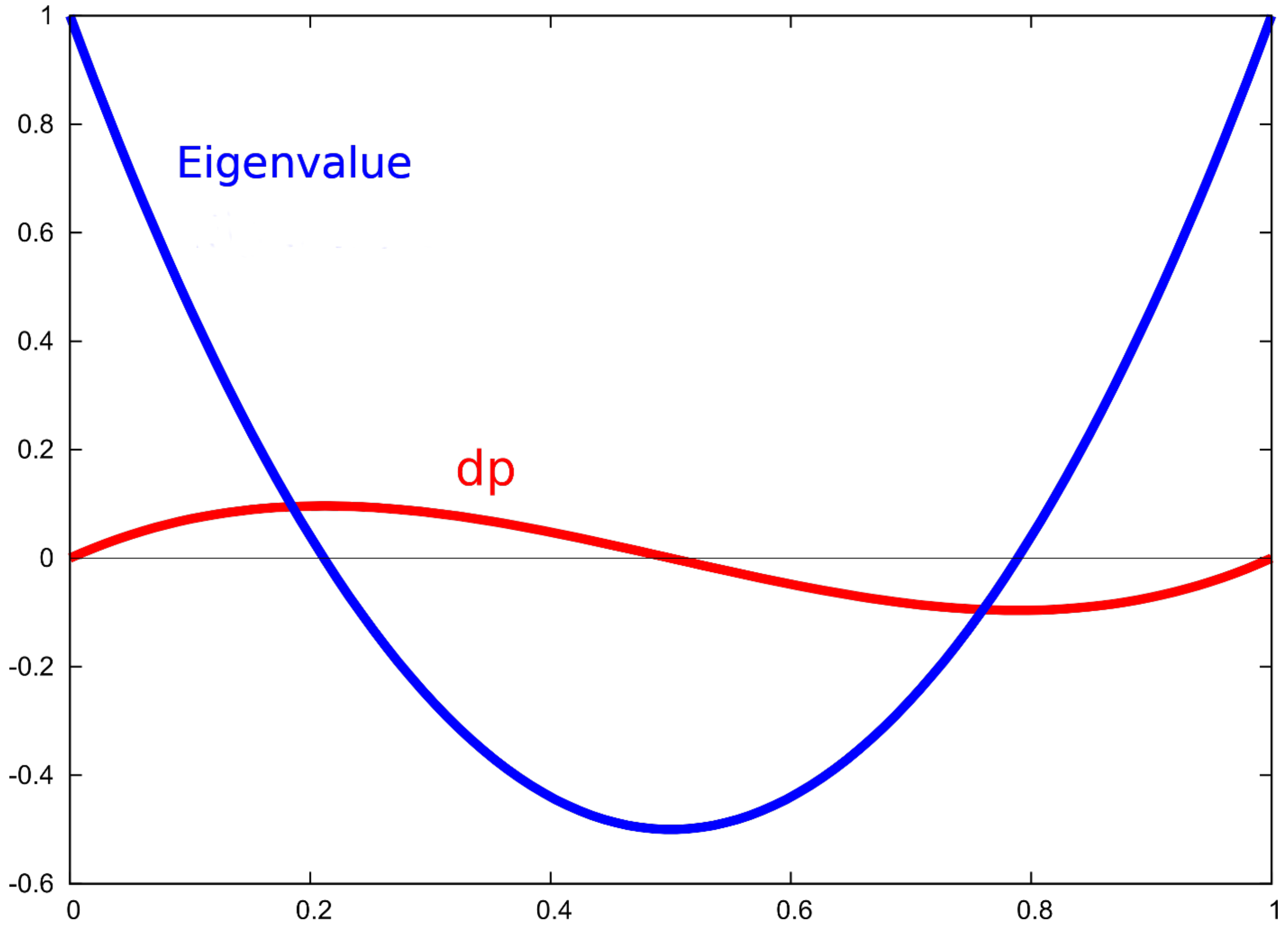
Example - Solution

- Linearising for solutions yields eigenvalues

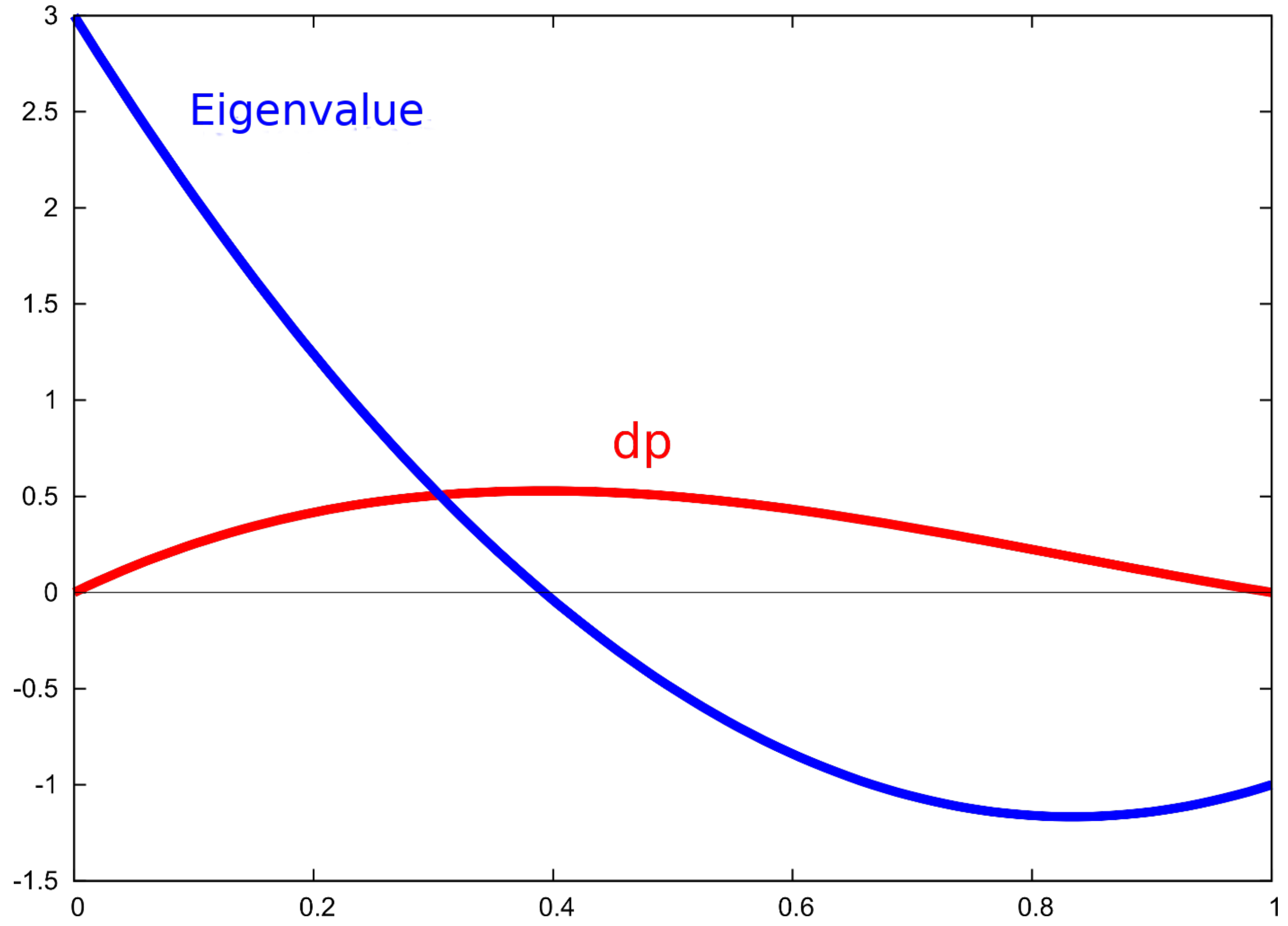
$$\lambda(p^*_{1,1}) = 1 \qquad \lambda(p^*_{1,2}) = 1 \qquad \lambda(p^*_{1,3}) = -0.5$$

- Thus, $p^*_{1,3}$ is stable, the other two solutions are not.

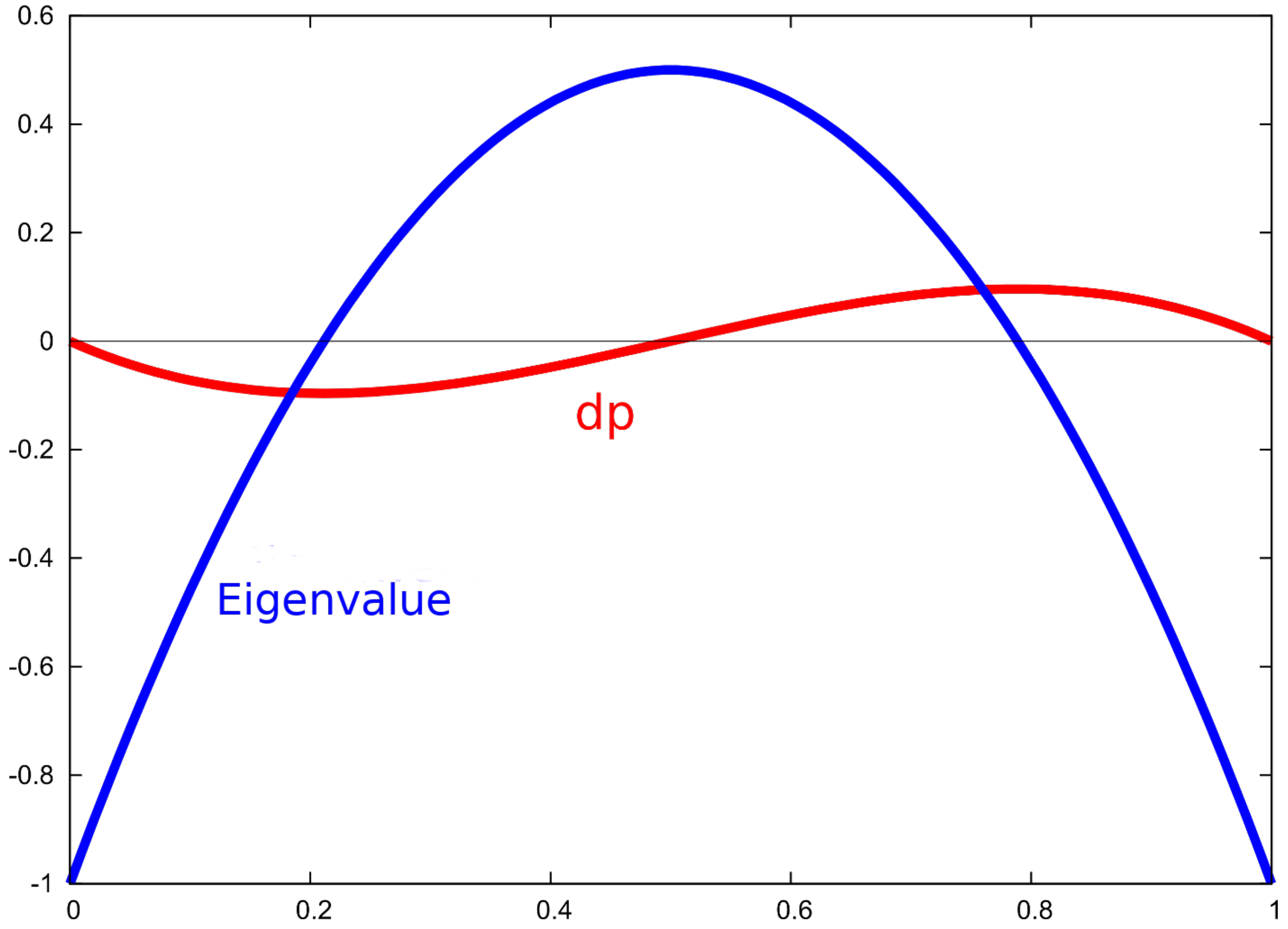
Dynamics: Anti-coordination games



Dynamics: Prisoners' dilemma



Dynamics: Coordination games



Constant populations

- The replicator dynamics example above uses population shares p_i as state variables in a standard replicator equation

$$\frac{dp_i}{dt} = p_i(f_i - \phi_t)$$

- Population shares sum to 1.0 at every time t; the development of population size is thus removed from the equation.

- The same holds for the other standard replicator equations

$$p_{i,t+1} = p_{i,t} + p_{i,t}(f_{i,t} - \phi_t)$$
$$p_{i,t+1} = p_{i,t}\frac{f_{i,t}}{\phi_t}$$

Growing populations: Example

- An industry sector produces with constant technology generating profits $x(t)$ at time t.
- A fixed share of the profits is reinvested and yields a constant growth rate $a = 0.1$.
- Demand is unlimited, therefore there is no exhaustion of growth or profits.
- How does the sector develop?

Growing populations: Example

- Defining the development in terms of population shares is not instructive as there is only one share.

- The state variable could be the profits $x(t)$.

- The replicator equation results as

$$\frac{dx}{dt} = 0.1x \qquad \text{general form:} \qquad \frac{dx}{dt} = ax$$

- The only equilibrium is $\frac{dx}{dt} = 0$, the eigenvalue is $\lambda = 0.1 > 0$ (general form $\lambda = a > 0$) not stable.

Growing populations: Example

- The replicator equation does, however, allow to derive the general sulution, the function $x(t)$ ...

$$\frac{1}{x}\frac{dx}{dt} = 0.1$$

$$\int \frac{1}{x}\frac{dx}{dt}dt = \int 0.1dt$$

$$\int \frac{1}{x}dx = \int 0.1dt$$

$$ln(x) = 0.1t + C'$$

$$x = e^{0.1t+C'} = e^{0.1t}e^{C'} = C_0e^{0.1t}$$

- The solution is

$$x(t) = C_0e^{0.1t} \qquad \text{general form:} \qquad x(t) = C_0e^{at}$$

Growing and shrinking populations

- Shrinking populations are modelled in the same way.
- If a population x shrinks at constant rate b
- ... we obtain the development equation $\frac{dx}{dt} = -bx$ with solution $x(t) = C_0 e^{-bt}$.
- Both effects combined would yield $\frac{dx}{dt} = (a-b)x$ with solution $x(t) = C_0 e^{(a-b)t}$.
- (Systems of this type are commonly used in evolutionary biology (with a and b being birth rate and death rate respectively).

Competition between non-constant populations

- Consider two populations $i = 1, 2$ with net growth rate a_i

$$\frac{dx_1}{dt} = a_1 x_1$$

$$\frac{dx_2}{dt} = a_2 x_2$$

- The equations are solved as

$$x_1(t) = C_1 e^{a_1 t}$$
$$x_2(t) = C_2 e^{a_2 t}$$

- Consider population shares $p_i = \frac{x_i}{\sum x}$

Competition between non-constant populations

- Population shares are obtained as

$$p_i(t) = \frac{C_i e^{a_i t}}{C_i e^{a_i t} + C_{j\neq i} e^{a_{j\neq j} t}}$$

- E.g. for p_1

$$p_1(t) = \frac{C_1 e^{a_1 t}}{C_1 e^{a_1 t} + C_2 e^{a_2 t}} = \frac{1}{1 + \frac{C_2}{C_1} e^{(a_2 - a_1)t}}$$

- ... which, for $t \to \infty$ shows the behavior:

$$\begin{array}{ll} lim_{t\to\infty} p_1(t) = 0 & \text{wenn } a_2 > a_1 \\ lim_{t\to\infty} p_1(t) = \frac{C_1}{C_1 + C_2} & \text{wenn } a_2 = a_1 \\ lim_{t\to\infty} p_1(t) = 1 & \text{wenn } a_2 < a_1 \end{array}$$

Capacity boundaries

- Growth may not be boundless, including in economic systems. Consider e.g. limited demand, linited labor force, limited energy, limited natural resources, limited ability to live off the planet without regard to climatic consequences.

- A simple way to model capacity boundaries is to set a boundary Z, such that growth slows down as the population approaches Z and reverses when Z is reached.

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{Z}\right)$$

- This system has equilibria $x_1^* = 0$ and $x_2^* = Z$

Capacity boundaries

- Stability analysis

$$\lambda = \frac{\partial \frac{dx}{dt}}{\partial x} = a - 2\frac{a}{Z}x$$

$$\lambda(x_1^*) = \lambda(0) = a > 0 \qquad \text{instabil}$$

$$\lambda(x_1^*) = \lambda(Z) = -a > 0 \qquad \text{stabil}$$

- The functional solution can also be derived, $x(t) = Z\frac{1}{1+C_0e^{-at}}$; since $lim_{t\rightarrow\infty}e^{-at} = 0$, the function converges towards the stable equilibrium $lim_{t\rightarrow\infty}x(t) = Z$ (except in case $C_0 = 0$).

Outlook

- We have now discussed methods that allow us to speak precisely about...
  - The relations among system components: *network theory*
  - The dynamics of systems: *dynamical systems theory*
  - The interaction among system components: *game theory*

- Combining concepts from game theory and dynamical systems has led us to consider evolutionary dynamics

- Using agent-based models allows us to bring all three approaches together

- Combining game theoretic and agent-based techniques is a powerful way to study interactions and institutions: It allows to relax many assumptions in creative ways (play multiple games, learn new strategies, etc., see the further readings)